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On the Generalized Euler–Frobenius Polynomial

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In this paper, the properties of the generalized Euler–Frobenius polynomial $\Pi_n(\cdot, q)$ are studied. It is proved that its zeroes are separated by the factor q and their asymptotic behavior, as $q \to \infty$, is given. As a consequence, it is shown that least squares spline approximation on a biinfinite geometric mesh can be bounded independently of the (local) mesh ratio q and that the norm of the inverse of the corresponding order k B-spline Gram matrix decreases monotonically to $2k - 1$ for large q, as $q \to \infty$.

1. INTRODUCTION

The exponential Euler polynomial $A_n(x; t)$ played an important role in the analysis of cardinal polynomial splines. This is much due to the fact that the spline defined by the functional relation

$$\begin{aligned} \phi_n(x) &:= A_n(x; \lambda), \qquad && x \in [0, 1[, \\ \phi_n(x+1) &:= \lambda \phi_n(x), && \text{otherwise,} \end{aligned}$$

vanishes at all integers for particular values of λ, the zeroes of the Euler–Frobenius polynomial $\Pi_n(\lambda) := (1 - \lambda)^n A_n(0; \lambda)$. A beautiful survey of

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cardinal polynomial splines can be found in [7]. Micchelli [6] showed that the essential properties of cardinal polynomial splines can be extended to the more general case of cardinal $\mathscr{L}$-splines. By applying his results to the particular differential operator

$$\mathscr{L}_t(D) := \prod_{i=0}^{n} (D - it), \qquad D := \frac{d}{dx}, \qquad t \in R$$

and to the corresponding generalized exponential Euler polynomial

$$A_n(x; \lambda, q) := \frac{1}{n!\, t^n} \sum_{i=0}^{n} (-)^{n-i} \binom{n}{i} \frac{q^{ix}}{q^i - \lambda}, \qquad q := e^t, \tag{1.1}$$

he analyzed spline interpolation at knots on the biinfinite geometric mesh

$$(q^i)_{-\infty}^{+\infty}. \tag{1.2}$$

In this case, the generalized Euler–Frobenius polynomial is given by

$$\Pi_n(\lambda; q) := \prod_{i=0}^{n} (q^i - \lambda) A_n(0; \lambda, q) = \frac{1}{n!\, t^n} \sum_{i=0}^{n} (-)^{n-i} \binom{n}{i} \prod_{\substack{j=0 \\ j \neq i}}^{n} (q^j - \lambda), \tag{1.3}$$

and satisfies a "difference-delay" equation [6]

$$\Pi_0(\lambda; q) := 1,$$

$$\Pi_{n+1}(\lambda; q) = \frac{1}{(n+1)\, t} \left((1 - \lambda)\, q^n \Pi_n(q^{-1}\lambda; q) - (q^{n+1} - \lambda)\, \Pi_n(\lambda; q)\right),$$

$$n = 0, 1, \ldots. \tag{1.4}$$

A recent paper by Höllig [5] shows that more general spline interpolation problems on a biinfinite geometric mesh can be understood in terms of properties of $\Pi_n(\lambda; q)$.

The main part of the present paper is an outline of some new characteristics of $\Pi_n(\lambda; q)$. A simple but far reaching property is the following. The zeroes $\mu_{n,i}(q)$ are separated by a factor q. This produces the bounds

$$-\text{const}_1\, q^{n-i} \leqslant \mu_{n,i}(q) \leqslant -\text{const}_2\, q^{n-i}$$

for some properly chosen positive const_1, const_2.

In Section 3, the properties developed are used in an analysis of spline interpolation Pf to f defined by the conditions

$$\int_I M_{i,r} Pf = \int_I M_{i,r} f, \qquad \text{all} \quad i,$$

on a biinfinite geometric mesh. In this way, some of the results in [5] are obtained by a different approach.

2. The Zeroes of $\Pi_n(\lambda; q)$

We start the section with the symmetries of the generalized Euler–Frobenius polynomial. In addition to the description (1.3), we shall use

$$\sum_{i=0}^{n-1} a_{n,i}(q)\,\lambda^i := \frac{1}{\gamma_n(q-1)^n}\,\Pi_n(\lambda; q), \qquad \gamma_n := \frac{1}{n!\,t^n},$$

to emphasize its polynomial character in λ.

Theorem 2.1. *The polynomial $\Pi_n(\lambda; q)$ satisfies*

$$\Pi_n(\lambda; q) = \lambda^{n-1} q^{-n(n-1)/2}\,\Pi_n(q^n\lambda^{-1}; q). \tag{2.1}$$

The coefficients $a_{n,i}(q)$ can be recurrently computed by

$$a_{n+1,i}(q) = (q-1)^{-1}((q^{n+1} - q^{n-i})\,a_{n,i}(q) + (q^{n+1-i} - 1)\,a_{n,i-1}(q)), \tag{2.2}$$

where

$$a_{n,0}(q) := 1, \qquad a_{n,-1}(q) = a_{n,n}(q) := 0.$$

Proof. For $n = 1$ or $\lambda = 0$, (2.1) obviously holds. Assume $\lambda \neq 0, n \geqslant 2$. Then

$$\begin{aligned}\Pi_n(\lambda; q) &= \gamma_n \sum_{i=0}^{n} (-)^{n-i} \binom{n}{i} \prod_{\substack{j=0\\ j\neq i}}^{n} (q^j - \lambda)\\ &= \gamma_n \sum_{i=0}^{n} (-)^{n-i} \binom{n}{i} q^i\lambda^{-1} \prod_{\substack{j=0\\ j\neq i}}^{n} (q^j - \lambda),\end{aligned}$$

since the nth order finite difference of a constant vanishes. But

$$q^i\lambda^{-1} \prod_{\substack{j=0\\ j\neq i}}^{n} (q^j - \lambda) = (-)^n\lambda^{n-1}q^{-n(n-1)/2} \prod_{\substack{j=0\\ j\neq n-i}}^{n} (q^j - q^n\lambda^{-1}),$$

which completes the proof of (2.1).

In terms of the $a_{n,i}(q)$, the recurrence relation (1.4) reads

$$\sum_{i=0}^{n} a_{n+1,i}(q)\,\lambda^i = -(q-1)^{-1}\left((1-\lambda)\,q^n \sum_{i=0}^{n-1} a_{n,i}(q)\,q^{-i}\lambda^i - (q^{n+1}-\lambda)\sum_{i=0}^{n-1} a_{n,i}(q)\,\lambda^i\right)$$

$$= (q-1)^{-1}\sum_{i=0}^{n}\left((q^{n+1}-q^{n-i})\,a_{n,i}(q) + (q^{n+1-i}-1)\,a_{n,i-1}(q)\right)\lambda^i,$$

if we define $a_{n,-1}(q) = a_{n,n}(q) := 0$, and this confirms (2.2). ∎

COROLLARY 2.1. *The coefficients $a_{n,i}(q)$ satisfy*

$$a_{n,i}(q) = q^{n(n-2i-1)/2}a_{n,n-1-i}(q), \tag{2.3}$$

and for $n \geqslant 2$

$$a_{n,i}(q) = q^{(n-i)(n-1-i)/2} \sum_{j=0}^{i(n-1-i)} a_{n,i}^{(j)} q^j. \tag{2.4}$$

The integer coefficients $a_{n,i}^{(j)}$ are symmetric

$$a_{n,i}^{(j)} = a_{n,i}^{(i(n-1-i)-j)}, \qquad \text{all} \quad j. \tag{2.5}$$

In particular,

$$a_{n,i}^{(0)} = \binom{n-1}{i},$$
$$a_{n,i}^{(1)} = (n-2)\binom{n-1}{i} - \binom{n-2}{i+1} - \binom{n-2}{i-2}. \tag{2.6}$$

It is easy to prove (2.3)–(2.6) by using (2.2) and mathematical induction. We shall omit this step.

From now on we think of the zeroes of $\Pi_n(\cdot\,; q)$ as functions of q. It is proved in [6] that the $n-1$ zeroes $\mu_{n,i}(q)$, $i = 1,..., n-1$, of $\Pi_n(\cdot\,; q)$ are all simple and real, in fact negative. They satisfy

$$\mu_{n,i}(q) < 0, \qquad \frac{d}{dq}\mu_{n,i}(q) < 0, \tag{2.7}$$

$$\lim_{q\to 0+} \mu_{n,i}(q) = 0, \qquad \lim_{q\to\infty} \mu_{n,i}(q) = -\infty, \qquad \text{all} \quad i, \tag{2.8}$$

and

$$\mu_{n,i}(q^{-1}) = \mu_{n,n-i}^{-1}(q), \qquad \text{all} \quad i. \tag{2.9}$$

We shall think of the $\mu_{n,i}(q)$ as ordered,

$$\mu_{n,1}(q) < \mu_{n,2}(q) < \cdots < \mu_{n,n-1}(q) < 0. \tag{2.10}$$

Then, additionally, by [4] and (2.15)

$$\frac{d}{dq}\left(\frac{\mu_{n,n-1}(q)}{q}\right) < 0, \qquad \frac{d}{dq}\left(\frac{\mu_{n,1}(q)}{q^{n-1}}\right) > 0. \tag{2.11}$$

The symmetry (2.9) tells us that we can restrict our discussion to the case $q \geqslant 1$.

LEMMA 2.1. *Let $q \geqslant 1$. Then*

$$\mu_{n,i-1}(q) < \mu_{n+1,i}(q) < q\mu_{n,i}(q), \quad i = 2, 3, ..., n-1;\ n = 2, 3, ... \tag{2.12}$$

Proof. Suppose $\mu_{n,i-1}(q) < q\mu_{n,i}(q)$ holds for some n. By hypothesis then

$$\operatorname{sign}(\Pi_n(q^{-1}\lambda; q)) \cdot \operatorname{sign}(\Pi_n(\lambda; q)) < 0, \lambda \in [q\mu_{n,i}(q), \mu_{n,i}(q)],$$

and from (1.4)

$$\Pi_{n+1}(\lambda; q) \neq 0, \qquad \lambda \in [q\mu_{n,i}(q), \mu_{n,i}(q)]. \tag{2.13}$$

But $\mu_{n,i}(q)$ is a zero of $\Pi_n(\cdot\,; q)$, thus another look at (1.4) tells us

$$\operatorname{sign}(\Pi_{n+1}(q\mu_{n,i}(q); q)) \cdot \operatorname{sign}(\Pi_{n+1}(\mu_{n,i-1}(q); q)) < 0,$$

and there is at least one zero of $\Pi_{n+1}(\cdot\,; q)$ in each of the intervals

$$]\mu_{n,i-1}(q), q\mu_{n,i}(q)[\,, \qquad \text{all} \quad i. \tag{2.14}$$

Also by (1.4)

$$\operatorname{sign}(\Pi_{n+1}(0+; q)) \cdot \operatorname{sign}(\Pi_{n+1}(\mu_{n,n-1}(q); q)) < 0,$$
$$\operatorname{sign}(\Pi_{n+1}(\mu_{n,1}(q); q)) \cdot \operatorname{sign}(\Pi_{n+1}(-\infty; q)) < 0,$$

and this reveals the position of the smallest and the largest zero of $\Pi_{n+1}(\cdot\,; q)$. However, $\Pi_{n+1}(\cdot\,; q)$ is a polynomial of degree $< n+1$, and in each of the intervals (2.14) there is exactly one zero, $\mu_{n+1,i}(q)$.

Now (2.15) brings the induction hypothesis to the next level and (2.12) is proved since it obviously holds for $n = 2$. ∎

It is easy to deduce the following interesting properties of $\mu_{n,i}(q)$.

COROLLARY 2.2. *The zeroes $\mu_{n,i}(q)$ of $\Pi_n(\cdot\,; q)$ have the following properties*

$$\mu_{n,i}(q) \cdot \mu_{n,n-i}(q) = q^n, \qquad \text{all} \quad i. \tag{2.15}$$

In particular

$$\mu_{2k,k}(q) = -q^k, \tag{2.16}$$

and for $i \leqslant \lfloor (n-1)/2 \rfloor$

$$\mu_{n,i}(q) < -q^{n-i}, \qquad \mu_{n,n-i}(q) > -q^i, \tag{2.17}$$

as well as

$$\frac{d}{dq}\left(\frac{\mu_{n,i}(q)}{q^n}\right) > 0, \qquad \text{all} \quad i. \tag{2.18}$$

Proof. By (2.1)

$$\Pi_n(\lambda; q) = 0 \qquad \text{iff} \quad \Pi_n(q^n/\lambda; q) = 0.$$

Since we have ordered $\mu_{n,i}(q)$ as in (2.10), (2.15) follows. Equation (2.16) is a special case of (2.15). From (2.15) and (2.12) we find

$$q^n = \mu_{n,i}(q)\, \mu_{n,n-i}(q) > q^{n-2i} \mu_{n,n-i}^2(q),$$

which implies (2.17). Finally, combining (2.15) and (2.7) we obtain (2.18). ∎

THEOREM 2.2. *Let $q \geqslant 1$. Then for $i = 1, 2, \ldots, \lfloor (n-1)/2 \rfloor$*

$$-c_1 q^{n-i} \leqslant \mu_{n,i}(q) \leqslant -c_2 q^{n-i}, \tag{2.19}$$

$$-\frac{1}{c_2} q^i \leqslant \mu_{n,n-i}(q) \leqslant -\frac{1}{c_1} q^i. \tag{2.20}$$

The constants c_1, c_2 do not depend on q and i, and

$$c_1 = |\mu_{n,1}(1)|, \tag{2.21}$$

$$1 < c_2 \leqslant \begin{cases} \dfrac{n+1}{n-1}, & n \text{ odd} \\[2ex] \dfrac{n+2}{n-2}, & n \text{ even} \end{cases} \tag{2.22}$$

Proof. It is enough to prove (2.19), since (2.20) follows from it by (2.15). Observe from (2.11) that $\mu_{n,1}(q) \geqslant \mu_{n,1}(1)\, q^{n-1}$. Then by Lemma 2.1

$$\mu_{n,i}(q) \geqslant \frac{\mu_{n,1}(q)}{q^{i-1}} \geqslant \mu_{n,1}(1)\, q^{n-i} = -c_1 q^{n-i},$$

and the left inequality is proved.

Since $\mu_{n,i}(q)/q^{n-i}$ is a continuous function on $[1, \infty[$ and satisfies (2.17), while by Theorem 2.4

$$\lim_{q\to\infty} \frac{\mu_{n,i}(q)}{q^{n-i}} = -\frac{n-i}{i},$$

there obviously exists a constant $1 < c_2 \leqslant \min_i (n-i)/i$ independent of q such that the right inequality of (2.19) holds. In particular note that $1 < c_2 \leqslant k/(k-1)$ for $n = 2k-1$. ∎

Theorem 2.2 bounds $\mu_{n,i}(q)$ as functions of q. However, it is of interest also to ask the opposite question: suppose $\mu_{n,i}(\tilde{q}) = \mu_{n,i-1}(q)$. What can we say about q, $\tilde{q}$? We believe its answer is beautiful enough to deserve its place in the paper.

THEOREM 2.3. *There exist a constant*, const < 1, *so that, for any* q, $\tilde{q}$ *or* i,

$$\mu_{n,i}(\tilde{q}) = \mu_{n,i-1}(q) \tag{2.23}$$

implies

$$q/\tilde{q} \leqslant \text{const} < 1.$$

Proof. Let q, $\tilde{q}$ satisfy (2.23) for some i, Then (2.18) gives us

$$\left(\frac{q}{\tilde{q}}\right)^n < \left(\frac{q}{\tilde{\tilde{q}}}\right)^n, \tilde{\tilde{q}} := \text{a solution of } \left(\frac{q}{\tilde{\tilde{q}}}\right)^n = \frac{\mu_{n,i}(q)}{\mu_{n,i-1}(q)} =: \rho_i(q).$$

The function $\rho_i(q)$ is a continuous function of q, and by Lemma 2.1 and (2.9)

$$q \geqslant 1 : \rho_i(q) < 1/q,$$

$$q < 1 : \rho_i(q) = \frac{\mu_{n,n-i+1}(1/q)}{\mu_{n,n-i}(1/q)} < q.$$

Thus $\rho_i(0+) = \rho_i(\infty) = 0$. Clearly we find

$$const = \max_i \max_q \rho_i(q) < 1. \quad ∎$$

The last part of this section we devote to the asymptotic behavior of $\mu_{n,i}(q)$ as $q \to \infty$.

THEOREM 2.4. *For* $i = 1, 2, \ldots, \lceil (n-1)/2 \rceil$,

$$\mu_{n,i}(q) = -\frac{n-i}{i} q^{n-i} - c_{n,i} q^{n-i-1} + O(q^{n-2-i}), \tag{2.24}$$

$$\mu_{n,n-i}(q) = -\frac{i}{n-i} q^{i} + \left(\frac{i}{n-i}\right)^2 c_{n,i} q^{i-1} + O(q^{i-2}). \tag{2.25}$$

Here

$$0 \leqslant c_{n,i} := \frac{1}{i^2(i+1)(n-1)(n-i+1)} [(n-2i)^4 + (6i-1)(n-2i)^3$$
$$+4i(3i-1)(n-2i)^2 + 4i^2(2i-1)(n-2i)]. \tag{2.26}$$

In particular,

$$c_{2k-1,k-1} = \frac{(2k-1)^2}{k(k-1)^2(k+1)}. \tag{2.27}$$

Proof. By (2.15), it is enough to prove (2.24). Let $\lambda = \mu_{n,i}(q)$. By (2.8) and (2.15)

$$\lim_{q \to \infty} \frac{\mu_{n,i}}{q^n} = 0, \qquad \text{all } i.$$

Thus for some $\alpha \neq 0$ and some $r > 0$

$$\lambda = \alpha q^{n-r} + \beta q^{n-r-1} + O(q^{n-r-2}). \tag{2.28}$$

Since the coefficients $a_{n,i}(q)$ are polynomials in q, and after a proper normalization in $1/q$, r is an integer. From Corollary 2.1 we conclude that as $q \to \infty$

$$\sum_{i=0}^{n-1} a_{n,i}(q) \lambda^i \approx \sum_{i=0}^{n-1} q^{(n+i)(n-1-i)/2 + (n-r)i}$$
$$\times (a_{n,i}^{(0)} \alpha^i + q^{-1}(a_{n,i}^{(1)} \alpha^i + a_{n,i}^{(0)} i \alpha^{i-1} \beta)). \tag{2.29}$$

An inspection of the exponent

$$\psi(i, r) := (n+i)(n-1-i)/2 + (n-r)\, i$$

shows that

$$\psi\left(\frac{2(n-r)-1}{2}+i, r\right)=\psi\left(\frac{2(n-r)-1}{2}-i, r\right),$$

$$\Delta_1 \psi(i, r) := \psi(i+1, r)-\psi(i, r)=n-r-i-1.$$

Since $\Delta_1 \psi(n-r-1, r)=0$, the leading power of q occurs in the terms $i=n-r-1$, $n-r$. Thus (2.29) can vanish precisely for $r=1, 2, \ldots, n-1$ as $q \to \infty$, and we conclude from (2.10): $\mu_{n,i}(q)=O(q^{n-i})$. By using (2.6), it is now straightforward to complete the proof. ∎

3. Polynomial Splines on a Biinfinite Geometric Mesh

To start more generally, let $\mathbf{t} := (t_i)_{-\infty}^{+\infty}$ be a strictly increasing biinfinite sequence with $t_{\pm\infty} := \lim_{i \to \pm\infty} t_i$, $I :=]t_{-\infty}, t_{+\infty}[$. Let further

$$mS_{n,\mathbf{t}}(I) := \{f : f \in c^{n-2}(I) \cap \mathbf{L}_\infty(I),\ f|_{]t_i, t_{i+1}[} \text{ is a polynomial of degree } < n\}$$

be the normed linear space of polynomial splines of order n with the breakpoint sequence $\mathbf{t}$ and the norm $\|f\| := \sup_{x \in I} |f(x)|$. Let $r, k \in \mathbf{N}$ be given integers, $0 \leqslant r < 2k$, $0 < k$. Consider the map

$$R_r : mS_{2k-r,\mathbf{t}}(I) \to l_\infty : f \mapsto (\phi_{i,r} f)_{-\infty}^{+\infty} \tag{3.1}$$

associated with interpolation conditions

$$\phi_{i,0} f := f(t_i), \qquad \phi_{i,r} f := \int_I M_{i,r} f, \qquad r > 0.$$

Here, as usual the B-splines of order k with knots $\mathbf{t}$ are defined by

$$M_{ik}(x) := k[t_i, t_{i+1}, \ldots, t_{i+k}](\cdot - x)_+^{k-1},$$

$$N_{ik} := \frac{1}{k}(t_{i+k}-t_i) M_{ik}.$$

The interpolation problem: for given $\mathbf{b} := (b_i)_{i=-\infty}^{+\infty} \in l_\infty$, find $f \in S_{2k-r,\mathbf{t}}(I)$ such that

$$R_r f = \mathbf{b}$$

is by [2] correct, if R_r is invertible, i.e., the Gramian (totally positive) matrix

$$G_r := (\phi_{i,r} N_{j,2k-r})_{i,j=-\infty}^{+\infty}$$

is boundedly invertible.

Let us restrict ourselves now to a particular geometric knot sequence $\mathbf{t} := (q^i)_{-\infty}^{+\infty}$ for some $q \in \,]0, \infty[$. In this case the matrix is a Toeplitz matrix and is boundedly invertible iff the characteristic polynomial

$$\sum_j \lambda^j \phi_{i,r} N_{j,2k-r} \tag{3.2}$$

has no zero on the unit circle $|\lambda| = 1$, or since G_r is totally positive, at $\lambda = -1$. The case $r = 0$ is treated in [6], where it is proved that

$$\Pi_{2k-1}(\lambda; q) = \sum_{j=0}^{2k-2} \lambda^j \phi_{2k-1,0} N_{j,2k} = \lambda^{2k-i-1} \sum_j \lambda^j \phi_{i,0} N_{j,2k}, \tag{3.3}$$

and from properties of the generalized Euler–Frobenius polynomial determined when R_0 is invertible. A nice argument shown to us by de Boor [3] leads to the conclusion: The characteristic polynomial (3.2) has -1 as a zero iff

$$\Pi_{2k-1}(-q^{k-r}; q) = 0$$

for any r, $0 \leqslant r \leqslant 2k-1$. A recent result of Höllig [5] states

$$\| G_r^{-1} \|_\infty = h_r(q) := \left| \frac{\Pi_{2k-1}(q^r; q)}{\Pi_{2k-1}(-q^r; q)} \right| . \tag{3.4}$$

He proves that $h_r(q)$ is bounded independently of q and G_r, $r \neq k-1, k$ is not boundedly invertible for at least one $q \in [1, \infty[$. We give here an alternative proof by simply rereading Theorem 2.2. By Theorem 2.1 we can restrict to the case $0 \leqslant r \leqslant k-1$.

The equation

$$\eta_i(q) := \eta_{i,k,r}(q) := \mu_{2k-1,i}(q)/q^r = -1 \tag{3.5}$$

has (at least one) solution $q \in \,]0, \infty[$ exactly for $r+1 \leqslant i \leqslant 2k-2-r$. Put

$$Q_r := \{q \mid q \text{ is a solution of } (3.5)\}$$

and $|Q_r| :=$ number of elements in Q_r. Choose $r+1 \leqslant i \leqslant k-1$. Then

$$\begin{aligned} -c_1 q^{2k-1-r-i} &\leqslant \eta_i(q) \leqslant -c_2 q^{2k-1-r-i}, \qquad && q \geqslant 1, \\ -c_2^{-1} q^{-r+i} &\leqslant \eta_i(q) \leqslant -c_1^{-1} q^{-r+i}, && q \leqslant 1. \end{aligned} \tag{3.6}$$

If $q \geqslant 1$ obviously there is no solution to (3.5) since this would imply $i \geqslant 2k-r$. In the case $q \leqslant 1$ there is $q \in Q_r$ exactly for $i \geqslant r+1$. Since

$$\eta_i(q) = -1 \qquad \text{iff} \quad \eta_{2k-1-i}(1/q) = -1$$

our claim is confirmed.

We note that the case of a finite partition [1] suggests that R_r, $r \neq k-1, k$ is not invertible for all q, but as already pointed out in [2] the same proof can not be applied since the quotients

$$\min_{i+1-r \leqslant j \leqslant i} \frac{q^{j+r} - q^j}{q^{i+1} - q^i}$$

are bounded independently of q (by r).

Let now $q \geqslant 1$. From (2.15) we get

$$h_k(q) = h_{k-1}(q) = \prod_{i=1}^{2k-2} \left| \frac{q^k - \mu_{2k-1,i}(q)}{q^k + \mu_{2k-1,i}(q)} \right| = \prod_{i=1}^{k-1} \frac{w_i(q)+1}{w_i(q)-1} \tag{3.7}$$

with

$$w_i(q) := -(\mu_{2k-1,i}(q) + \mu_{2k-1,2k-1-i}(q))/(q^k + q^{k-1}). \tag{3.8}$$

From Theorem 2.2 we conclude

$$\bar{w}_i(q) := -\mu_{2k-1,i}(q)/q^k \geqslant w_i(q) \geqslant (c_2 q^{k-1-i} + c_2^{-1} q^{-k+1+i})/2 =: \mathbf{w}_i(q)$$

and

$$\bar{h}_{k-1}(q) := \prod_{i=1}^{k-1} \frac{\bar{w}_i(q)+1}{\bar{w}_i(q)-1} \leqslant h_{k-1}(q) < \mathbf{h}_{k-1}(q) := \prod_{i=1}^{k-1} \frac{\mathbf{w}_i(q)+1}{\mathbf{w}_i(q)-1}.$$

Since $\mathbf{h}_{k-1}(q)$ is decreasing as a function of q, this suggests that $h_{k-1}(q)$ is too. However, we succeeded in proving this only as $q \to \infty$, as a consequence of Theorem 2.4 and (3.7), (3.8).

THEOREM 3.5. *For $0 \leqslant r \leqslant 2k-1$, the Gramian matrix G_r is not boundedly invertible for $q \in Q_r$, and $|Q_r| = |Q_{2k-1-r}| \geqslant 2(k-1-r)$, $0 \leqslant r \leqslant k-1$. In particular, $|Q_k| = |Q_{k-1}| = 0$, and the norm $h_k(q)$ for $q \in [1, \infty[$ satisfies*

$$\bar{h}_k(q) \leqslant h_k(q) < \mathbf{h}_k(q) < \left(\frac{c_2+1}{c_2-1}\right)^{2(k-1)}; \tag{3.9}$$

also as $q \to \infty$

$$h_k(q) = (2k-1)(1 + 4(k-2)/(k+1)\, q^{-1} + O(q^{-2})). \tag{3.10}$$

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